

## EQUIVALENT BIRATIONAL EMBEDDINGS II: DIVISORS

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ABSTRACT. Two divisors in  $\mathbb{P}^n$  are said to be Cremona equivalent if there is a Cremona modification sending one to the other. We produce infinitely many non equivalent divisorial embeddings of any variety of dimension at most 14. Then we study the special case of plane curves and rational hypersurfaces. For the latter we characterise surfaces Cremona equivalent to a plane, under a mild assumption.

## INTRODUCTION

Let  $X_1, X_2 \subset \mathbb{P}^n$  be two birationally equivalent projective varieties. It is natural to ask if there exists a Cremona transformation of  $\mathbb{P}^n$  that maps  $X_1$  to  $X_2$ , in this case we say that  $X_1$  and  $X_2$  are Cremona equivalent, see Definition 1.1 for the precise statement. This is somewhat related to the Abhyankar–Moh problem, [AM] and [Je]. Quite surprisingly the main theorem in [MP] states that this is the case as long as the codimension of  $X_i$  is at least 2. In this work we want to study the case of divisors. It is easy to give examples of pairs of birationally equivalent divisors that are not Cremona equivalent. Our approach is to use Log Minimal Model Program, LMMP for short, and its variants like Sarkisov Theory and Noether Fano inequalities. Via these techniques we are able to produce many examples of these inequivalent embeddings in arbitrary dimensions. Moreover we prove that any irreducible and reduced variety, of dimension at most 14, admits infinitely many non Cremona equivalent divisorial embeddings. This shows the difficulty of classifying inequivalent embeddings. This is why we next concentrate on two special classes of divisors: plane curves and rational hypersurfaces. One can look at Cremona equivalence as the action of the Cremona group of  $\mathbb{P}^n$  on the Hilbert scheme of divisors. For  $\mathbb{P}^2$  both Cremona group and Sarkisov theory are well understood. This allows us to address a question posed in [Ii] about minimal degree curves. In Theorem 3.16 we give a necessary and sufficient condition to be a curve of minimal degree under the Cremona equivalence. This description is not fully satisfactory. We do not have a straightforward way to decide whether a curve is minimal or not without a partial resolution of the singularities. On the other hand examples, see Example 3.19, show that nested infinitely near singularities give unpredictable behaviour with respect to the Cremona action. The second class of divisors we consider is that of rational hypersurfaces. A nice result proposed by Coolidge state that a plane curve is Cremona equivalent to a line if and only if  $\bar{\kappa}(\mathbb{P}^2, C) < 0$ . This statement has been proved and somewhat strengthened by Kumar and Murthy,

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[KM]. Our approach with LMMP techniques gives a new proof of it and suggests a possible extension in arbitrary dimension. The idea is to consider a log resolution, say  $(S, \overline{C})$ , of the pair  $(\mathbb{P}^2, C)$  and translate the hypothesis on Kodaira dimension into a geometric restriction to the possible contractions occurring along a LMMP directed by  $\overline{C}$ . In this way we end up on log varieties we are able to control. This reminded us the  $\sharp$ -MMP, [Me], where again numerical constrain were used to control the birational modification occurring along a LMMP. With these two constructions in mind we are able to prove a Coolidge type statement also for rational surfaces in  $\mathbb{P}^3$ , Theorem 4.13 and in a weaker form for arbitrary rational divisors in  $\mathbb{P}^n$ , Remark 4.17.

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After this paper was completed Ciro Ciliberto brought to our attention his work with Alberto Calabri, [CC], devoted to a detailed study of minimal degree plane curves.

## 1. NOTATIONS AND PRELIMINARIES

We work over an algebraically closed field of characteristic zero. We are interested in Birational transformations of log pairs. For this we introduce the following definition.

**Definition 1.1.** Let  $D \subset X$  be an irreducible and reduced divisor on a normal variety  $X$ . We say that  $(X, \alpha D)$  is birational to  $(X', \alpha D')$ , for  $\alpha \in \mathbb{Q}$ , if there exists a birational map  $\varphi : X \dashrightarrow X'$  with  $\varphi_*(D) = D'$ , in particular  $\varphi$  is defined on a generic point of  $D$ . Let  $D \subset \mathbb{P}^n$  be an irreducible reduced divisor then we say that  $D$  is Cremona equivalent to  $D'$  if  $(\mathbb{P}^n, D)$  is birational to  $(\mathbb{P}^n, D')$ .

Let us proceed recalling a well known class of singularities.

**Definition 1.2.** Let  $X$  be a normal variety and  $D = \sum d_i D_i$  a  $\mathbb{Q}$ -Weil divisor, with  $d_i \leq 1$ . Assume that  $(K_X + D)$  is  $\mathbb{Q}$ -Cartier. Let  $f : Y \rightarrow X$  be a log resolution of the pair  $(X, D)$  with

$$K_Y = f^*(K_X + D) + \sum a(E_i, X, D)E_i$$

We call

$$disc(X, D) := \min_{E_i} \{a(E_i, X, D) \mid E_i \text{ is an } f\text{-exceptional divisor for some log resolution}\}$$

Then we say that  $(X, D)$  is

$$\left. \begin{array}{l} \text{terminal} \\ \text{canonical} \end{array} \right\} \quad \text{if } disc(X, D) \quad \left\{ \begin{array}{l} > 0 \\ \geq 0 \end{array} \right.$$

While working on plane curves we frequently use ruled surfaces to fix the notations we recall the following definition.

**Definition 1.3.** Let  $\mathbb{F}_a := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-a))$  be the Hirzebruch surface. Then the exceptional section ( a fibre for  $a = 0$ ) will be called  $C_0$  and a fibre  $f$ .

In the final section we will be using some standard Cremona maps of  $\mathbb{P}^3$ , see for instance [SR], and birational modifications of scrolls. We find it convenient to group them here.

*Construction 1.4* ( $T_{2,3}$ ). Let  $l \subset \mathbb{P}^3$  be a line and  $p_1, p_2, p_3$  three general points. The linear system of quadrics through this configuration is homaloidal and gives rise to a Cremona transformation of type  $T_{2,3}$ . For our purpose the following facts are important:

- the rational normal curves secant to  $l$  and passing through the  $p_i$ 's are sent to lines;
- the plane  $P$  spanned the  $p_i$ 's is contracted to a line by the linear system of conics through the  $p_i$ 's and the point of intersection  $l \cap P$ .

*Construction 1.5* ( $T_{3,3}$ ). Let  $\Gamma \subset \mathbb{P}^3$  be a rational normal curve. Two general cubics containing  $\Gamma$  intersect along  $\Gamma \cup C$ . Then the linear system of cubics through  $C$  is homaloidal and produces a Cremona transformation of type  $T_{3,3}$ . It is a pleasant exercise to check that any rational cubic with isolated singularities is contained in one such linear system, and is therefore Cremona equivalent to a plane.

**Definition 1.6.** Let  $X := \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^1$  be a scroll, and  $x \in F \subset X$  a point. Then the elementary transformation centred at  $x$  is

$$\text{elm}_x : X \dashrightarrow X'$$

the composition of the blow up of  $x$  and the contraction of the strict transform of  $F$ . Then  $X'$  is still a scroll over  $\mathbb{P}^1$ .

Let  $X := \mathbb{P}(\mathcal{E}) \rightarrow W$  be a scroll over a surface, and  $\Gamma \subset X$  a smooth curve section. Let  $\pi : X \rightarrow W$  be the scroll structure and  $F := \pi^*(\pi_*(\Gamma))$ . Then the elementary transformation centred at  $\Gamma$  is

$$\text{elm}_\Gamma : X \dashrightarrow X'$$

the composition of the blow up of  $\Gamma$  and the contraction of the strict transform of  $F$ . Then  $X'$  is still a scroll over  $W$ .

*Construction 1.7.* A nice feature of these elementary transformations is the following. Let  $X$  be a 3-fold scroll over  $W$ , and  $S \subset X$  a smooth surface section. Then  $\pi|_S : S \rightarrow W$  is a birational map. Let  $D \subset X$  be a general surface section and  $\Gamma := S \cap D$ . Then we have  $\text{elm}_\Gamma(S) \cong W$ .

## 2. EXISTENCE RESULTS

We expect that any projective variety has infinitely many non Cremona equivalent divisorial embeddings. A slight variation of [MP, Lemma 3.1] together with the remarkable work of Mather, [Ma1], [Ma2], [Ma3] allows us to prove it under a dimensional bound.

**Theorem 2.1.** *Let  $X$  be an irreducible reduced projective variety of dimension  $k$ . Assume that  $k \leq 14$  then there are infinitely many non Cremona equivalent embeddings of  $X$  in  $\mathbb{P}^{k+1}$ .*

To prove the theorem we recall in a slightly generalised form [MP, Lemma 3.1].

**Lemma 2.2.** *Let  $X^{n-1}$  be an irreducible and reduced projective variety. Let  $\mathcal{L}$  and  $\mathcal{G}$  birational embeddings of  $X$  into  $\mathbb{P}^n$ , of degree respectively  $l$  and  $g$ . Assume that*

$l > g$  and  $\varphi_{\mathcal{L}}(X)$  is Cremona equivalent to  $\varphi_{\mathcal{G}}(X)$ . Then the pair  $(\mathbb{P}^n, \frac{n+1}{l}\varphi_{\mathcal{L}}(X))$  has not canonical singularities.

*Proof.* Let  $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$  be a Cremona equivalence between  $\varphi_{\mathcal{L}}(X)$  and  $\varphi_{\mathcal{G}}(X)$ . Fix a resolution of  $\Phi$

$$\begin{array}{ccc} & Z & \\ p \swarrow & & \searrow q \\ \mathbb{P}^n & \xrightarrow{\Phi} & \mathbb{P}^n \end{array}$$

Then we have

$$\mathcal{O}_Z \sim p^*(\mathcal{O}(K_{\mathbb{P}^n} + \frac{n+1}{l}\varphi_{\mathcal{L}}(X))) = K_Z + \frac{n+1}{l}X_Z - \sum a_i E_i$$

and

$$q^*(\mathcal{O}(K_{\mathbb{P}^n} + \frac{n+1}{l}\varphi_{\mathcal{G}}(X))) = K_Z + \frac{n+1}{l}X_Z - \sum b_i F_i$$

where  $E_i$ , respectively  $F_i$ , are  $p$ , respectively  $q$ , exceptional divisors. Let  $l \subset \mathbb{P}^n$  be a general line in the right hand side  $\mathbb{P}^n$ . Then the hypothesis  $l > g$  yields

$$0 > q^{-1}l \cdot (q^*(\mathcal{O}(K_{\mathbb{P}^n} + \frac{n+1}{l}\varphi_{\mathcal{G}}(X))) + \sum b_i F_i) = (\sum a_i E_i) \cdot q^{-1}l$$

This proves that at least one  $a_i < 0$  proving the claim.  $\square$

*Proof of Theorem 2.1.* Let  $X^k$  be an irreducible reduced projective variety and  $\mathcal{A}$ ,  $\mathcal{B}$  very ample linear system of degree at least  $(k+1)(k+2)$ . Consider two general sublinear system  $\mathcal{L} \subset \mathcal{A}$  and  $\mathcal{G} \subset \mathcal{B}$  both of projective dimension  $k+1$ . Let  $X_1 = \varphi_{\mathcal{L}}(X) \subset \mathbb{P}^{k+1}$  and  $X_2 = \varphi_{\mathcal{G}}(X) \subset \mathbb{P}^{k+1}$ . Lemma 2.2 shows that  $X_1$  and  $X_2$  are not Cremona equivalent as long as  $\deg X_1 > \deg X_2$  and

$$\max_{x \in X_1} \{\text{mult}_x X_1\} \leq \frac{\deg X_1}{k+2}$$

If  $k = \dim X \leq 14$  then by Mather transversality [Ma1] [Ma3], see also [BE], a general projection has points of multiplicity at most  $k+1$  and therefore  $X_1$  is not Cremona equivalent to  $X_2$ .  $\square$

*Remark 2.3.* Mather's results are optimal as explained by Lazarsfeld, [La, Theorem 7.2.19] see also [BE]. In general it is not known a bound, depending only on the dimension, for the singularities of a general projection.

Via Lemma 2.2 it is easy to give examples of inequivalent embeddings in arbitrary dimension. Consider a smooth codimension two subvariety  $X_{a,b} \subset \mathbb{P}^{k+2}$  given by the complete intersection of a form of degree  $a$  with a form of degree  $b$ , for  $a \leq b$ . Then a general projection to  $\mathbb{P}^{k+1}$  has degree  $ab$  and points of multiplicity at most  $a$ . On the other hand projecting from a general point of  $X$  produces a birational embedding of  $X$  of degree  $ab-1$ . By Lemma 2.2 these two embeddings are not birational equivalent as long as  $b \geq k+2$ . Theorem 2.1 and this example show that the classification of inequivalent embeddings is almost hopeless for a general variety. On the other hand there are special classes of varieties for which something more can be said.

## 3. PLANE CURVES

In this section we study the Cremona equivalence for plane curves. Our aim is to describe minimal degree representative in each class of Cremona equivalence.

**Definition 3.1.** Let  $C \subset S$  be an irreducible reduced curve on a smooth surface  $S$ . Then  $C$  is a Cartier divisor and to any valuation  $\nu$  of  $K(S)$  we associate the multiplicity  $\text{mult}_\nu C$ . Let  $\text{Sing}(C) = \{p_i\}$  be the set of valuations with multiplicity  $m_i = \text{mult}_{p_i} C > 1$ , and  $\text{mult}(C) = \{m_i\}$  the associated set of multiplicities. We always order the  $p_i$  in such a way that  $m_i \geq m_{i+1}$ .

*Remark 3.2.* The centres of valuations  $p_i$  are always points, on every birational model of  $S$  this is why we opted for the notation  $p_i$ , and frequently call them points, if no confusion is likely to arise.

It is clear that any pair  $(\mathbb{P}^2, C)$  is birational to a pair  $(\mathbb{F}_a, \tilde{C})$ . Our first aim is to choose a nice representative of the pair  $(\mathbb{P}^2, C)$ . This is a slight variation on the usual Sarkisov program for log surfaces, [BM].

**Proposition 3.3.** *Let  $C \subset \mathbb{P}^2$  be an irreducible reduced curve of degree  $d$ . Then  $(\mathbb{P}^2, C)$  is birational to one of the following:*

$(\mathbb{P}^2, l) \quad l \subset \mathbb{P}^2 \text{ a line}$	$(\mathbb{P}^2, C) \quad \begin{aligned} C &\sim \mathcal{O}(d') \\ K_{\mathbb{P}^2} + \frac{3}{d'}C &\equiv 0 \\ (\mathbb{P}^2, \frac{3}{d'}C) &\text{ terminal} \end{aligned}$
$(\mathbb{F}_0, \overline{C}) \quad \begin{aligned} \overline{C} &\sim \alpha f_0 + \beta f_1 \\ K_{\mathbb{F}_0} + \frac{2}{\alpha}\overline{C} &\text{ nef} \\ (\mathbb{F}_0, \frac{2}{\alpha}\overline{C}) &\text{ terminal} \end{aligned}$	$(\mathbb{F}_a, \overline{C}) \quad \begin{aligned} \overline{C} &\sim \alpha C_0 + \beta f_1 \\ K_{\mathbb{F}_a} + \frac{2}{\alpha}\overline{C} &\text{ nef} \\ (\mathbb{F}_a, \frac{2}{\alpha}\overline{C}) &\text{ canonical} \\ (\mathbb{F}_a, \frac{2}{\alpha}\overline{C}) &\text{ terminal along } C_0 \end{aligned}$

*Proof.* We prove the statement by induction on the degree of  $C$ . Assume that  $(\mathbb{P}^2, \frac{3}{d}C)$  is not terminal hence

$$(1) \quad m_1 \geq \frac{d}{3}$$

If  $d - m_1 = 1$ , then  $C$  is Cremona equivalent to a line.

Assume that  $d - m_1 \geq 2$  and let  $\epsilon : \mathbb{F}_1 \rightarrow \mathbb{P}^2$  be the blow up of  $p_1$ . Let

$$\overline{C} = \epsilon_*^{-1}C \sim (d - m_1)C_0 + df$$

be the strict transform. Then

$$\overline{C} \cdot C_0 = m_1$$

and we have

$$(K_{\mathbb{F}_1} + \frac{2}{d - m_1}\overline{C}) \cdot C_0 = -1 + \frac{2m_1}{d - m_1} \geq 0$$

In particular

$$K_{\mathbb{F}_1} + \frac{2}{d - m_1}\overline{C} \text{ is nef}$$

Let  $\phi : \mathbb{F}_1 \dashrightarrow \mathbb{F}_b$  be a sequence of elementary transformations centered on non canonical points and non terminal points along  $C_0$ .

To simplify notations we will call  $\overline{C}$  all strict transform of  $C$  along this chain of elementary transformations. Let us understand better this process. Let  $p \in \overline{C}$  be

a point of multiplicity  $m$ . Let  $\phi : \mathbb{F}_a \dashrightarrow \mathbb{F}_{a\pm 1}$  be the elementary transformation centered at  $p$ . If  $p \in C_0$  then the elementary transformation moves the singularity away from the exceptional section. Note that  $\phi(\overline{C})$  has a new point  $q$  of multiplicity

$$m_q = d - m_1 - m.$$

In particular if we have  $m \geq \frac{d-m_1}{2}$  then  $m_q \leq \frac{d-m_1}{2}$ .

This means that after finitely many elementary transformations we obtain a pair

$$(\mathbb{F}_b, \frac{2}{d-m_1}\overline{C}) \quad \text{canonical and terminal in a neighbourhood of } C_0.$$

Let

$$\overline{C} \sim (d - m_1)C_0 + \beta f$$

If  $b \geq 2$  we have finished because  $K_{\mathbb{F}_b} \cdot C_0 \geq 0$ . Assume that  $b = 1$  and

$$K_{\mathbb{F}_1} + \frac{2}{d-m_1}\overline{C} \quad \text{is not nef}$$

then

$$\frac{2}{d-m_1}(m_1 - d + \beta) < 1$$

that is

$$2\beta < 3(d - m_1)$$

By equation (1) this yields

$$\beta < d.$$

In other words  $C$  is birational to a plane curve of degree  $\beta < d$  and we conclude by induction on the degree. Assume finally that  $b = 0$  and

$$K_{\mathbb{F}_0} + \frac{2}{d-m_1}\overline{C} \quad \text{is not nef}$$

then  $\beta < (d - m_1)$ . This time we change the coefficient and repeat the argument for  $(\mathbb{F}_0, \frac{2}{\beta}\overline{C})$ . The coefficients used are in  $\frac{2}{\mathbb{N}}$ , therefore after finitely many steps we find the required model.  $\square$

**Definition 3.4.** Let  $C \subset \mathbb{P}^2$  be a curve. A standard model of  $(\mathbb{P}^2, C)$  is a birational pair  $(\mathbb{F}_a, \overline{C})$  obtained via the construction of Proposition 3.3.

The main difficulty in using these models is that they are not always unique.

*Remark 3.5.* Thanks to the uniqueness of minimal models for surfaces the models with terminal singularities are unique. Note that in presence of canonical singularities the uniqueness is lost. Let  $C \subset \mathbb{P}^2$  be a 6-ic curve with an ordinary double point and a tac-node. If we blow up the ordinary double point we end up with a model  $(\mathbb{F}_1, \frac{1}{2}C)$ . While resolving the tac-node we produce a model  $(\mathbb{F}_2, \frac{1}{2}C)$ .

Even if not unique the above models allow us to choose minimal degree representative in every Cremona class. Let us start with the following.

**Proposition 3.6.** *Let  $(S, C)$  and  $(S', C')$  be two birational, not biregular, models in the list of Proposition 3.3. Then both  $S$  and  $S'$  are ruled surfaces. Let  $\Phi : (S, C) \dashrightarrow (S', C')$  be a birational map of the pairs and assume that  $C \sim \alpha C_0 + \beta f$  and  $C' \sim \alpha' C_0 + \beta' f$ . Then*

- i)  $\alpha = \alpha'$ ;

- ii) if  $\kappa(S, 2/\alpha C) = 1$ , then  $\Phi$  is an isomorphism on the generic fibre of the ruled surfaces.

*Proof.* All our model have canonical singularities therefore the log-Kodaira dimension is preserved. Note that the only element with negative Kodaira is the first listed. We can assume that  $\kappa(S, 2/\alpha C) = \kappa(S', 2/\alpha' C') \geq 0$ . We already noticed that two terminal models are isomorphic. Hence the models involving  $\mathbb{P}^2$  are unique.

Assume that  $\alpha' \geq \alpha$  then  $(S, 2/\alpha' C)$  has canonical singularities therefore

$$\kappa(S, 2/\alpha' C) = \kappa(S', 2/\alpha' C') \geq 0.$$

On the other hand every model with non negative Kodaira dimension has infinitely many curves  $Z_\lambda$  such that

$$(K_S + 2/\alpha C) \cdot Z_\lambda = 0.$$

This yields  $\alpha = \alpha'$ . Assume that  $\kappa(S, 2/\alpha C) = 1$ , then  $S \cong \mathbb{F}_a$  and  $S' \cong \mathbb{F}_{a'}$ . Then the fibre structure on  $\mathbb{F}_a$  is the log-Iitaka fibration of the pair. Therefore it is preserved by any birational map of the pair.  $\square$

*Remark 3.7.* Note that Proposition 3.6 (ii) is false for pairs with zero log-Kodaira dimension. Consider a sextic  $C$  with at least four ordinary double points. Then blowing up the fourth node realizes a model but one can equivalently apply a standard Cremona transformation on the first three nodes and then blow up the fourth.

*Remark 3.8.* Let  $D \sim aC_0 + bf$  be an irreducible curve in  $\mathbb{F}_a$ . Let  $elm_p : \mathbb{F}_a \dashrightarrow \mathbb{F}_{a \pm 1}$  be the elementary transformation based on a point  $p$  of multiplicity  $m$  for the curve  $D$ . Let  $D' = elm_{p*}(D) \sim aC_0 + b'f$  be the strict transform. Then according to the position of  $p$  with respect to  $C_0$  we have

$$b' = b - m \text{ if } p \notin C_0, \quad b' = b + (a - m) \text{ if } p \in C_0$$

**Definition 3.9.** An irreducible and reduced curve  $C \subset \mathbb{P}^2$  is a minimal degree curve if it is not Cremona equivalent to any curve of lower degree.

We are ready to put the first brick.

**Lemma 3.10.** Let  $C \subset \mathbb{P}^2$  be a curve of degree  $d > 1$ , with multiplicity set  $\{m_i\}$ . Assume the following

- a)  $m_1 > d/3$
- b)  $C$  is a minimal degree curve
- c)  $(S, \overline{C})$  is a model of  $(\mathbb{P}^2, C)$ , with  $\overline{C} \sim \alpha C_0 + \beta f$
- d)  $\kappa((S, 2/\alpha \overline{C})) = 1$

Then  $\alpha = d - m_1$ ,  $S \cong \mathbb{F}_a$ , and the general fibre of  $\mathbb{F}_a$  is the strict transform of a line through  $p_1$ .

*Remark 3.11.* In particular the pair  $(\mathbb{P}^2, 3/dC)$ , as in the statement of Lemma 3.10, has at most two places of non canonical singularities on  $\mathbb{P}^2$ . If moreover the places are two then all other singularities are terminal, again the assumption on Kodaira dimension is crucial. We would like to thank Alberto Calabri and Ciro Ciliberto for pointing this out to us while we were cruising in wrong directions.

*Proof.* Let  $\nu : \mathbb{F}_1 \rightarrow \mathbb{P}^2$  be the blow up of  $p_1$ . Then

$$\nu_*^{-1}(C) = \overline{C} \sim (d - m_1)C_0 + df$$

If either  $m_2 < \frac{d-m_1}{2}$  or  $m_2 > \frac{d-m_1}{2}$  and  $m_3 < \frac{d-m_1}{2}$  we produce a terminal model, therefore unique, and the claim is clear.

If  $m_2 > \frac{d-m_1}{2}$  and  $m_3 \geq \frac{d-m_1}{2}$  then assumption b) forces all non terminal points to lie on  $C_0$ . Let  $(\mathbb{F}_a, \tilde{C})$  be a standard model of  $(\mathbb{P}^2, C)$ . Let  $\tilde{C} \sim \alpha C_0 + \tilde{\beta} f$ , then we have.

*Claim.* The general fibre of  $\mathbb{F}_a$  is a line through  $p_1$  and  $\alpha = d - m_1$ .

*Proof of the claim.* The standard model  $(\mathbb{F}_a, \tilde{C})$  is obtained via a chain of elementary transformations, starting from the blow up of a point of maximal multiplicity of  $C$ . In particular the general fibre, after this blow up, is a line passing through  $p_1$ . The only ruled surface with two fibre structures is  $\mathbb{F}_0$ . Hence to change the general fibre the chain of elementary transformations has to go back to  $\mathbb{F}_0$ , and henceforth to  $\mathbb{F}_1$ . Let  $\psi : \mathbb{F}_a \dashrightarrow \mathbb{F}_1$  such a map, and

$$\overline{C} := \psi_*(\tilde{C}) \sim (d - m_1)C_0 + \overline{d}f$$

be the strict transform of  $C$  on  $\mathbb{F}_1$ . Then by Remark 3.8 we have

$$\overline{d} = d + \sum_1^k ((d - m_1) - \mu_j^0) - \sum_1^k \mu_h$$

where  $\mu_j^0$  are the multiplicities of points successively lying on the exceptional section and  $\mu_h$  are the multiplicities of points lying outside of it. Note that by construction both  $\mu_h$  and  $\mu_j^0$  are bounded from below by  $\frac{d-m_1}{2}$ . Furthermore by hypothesis we have  $\mu_1^0 = m_2 > \frac{d-m_1}{2}$ . Therefore the existence of such a chain produces a plane curve of degree  $d' < d$  Cremona equivalent to  $C$ .  $\square$

The claim and Proposition 3.6(ii) allow to conclude in this case.

Assume now that  $m_2 = \frac{d-m_1}{2}$ . Let  $(\mathbb{F}_a, \tilde{C})$  be a standard model. Then we only blow up points on the exceptional section. In particular no chain of elementary transformation can land on  $\mathbb{F}_0$ . This is enough to conclude again by Proposition 3.6(ii).  $\square$

*Remark 3.12.* The proof of Lemma 3.10 also yields the following. Let  $C_d \subset \mathbb{P}^2$  be a minimal degree curve, with  $m_1 \geq d/3 > 1$ . Then a standard model of  $(\mathbb{P}^2, C_d)$  is a ruled surface.

**Definition 3.13.** Let  $\Sigma \subset \mathbb{F}_a$  be a (eventually reducible) section,  $f_1$  and  $f_2$  two fibres and  $\tilde{\Sigma} \sim \Sigma + f$  an irreducible section. The linear system

$$\Lambda_\Sigma = \{\Sigma + f_1, \Sigma + f_2, \tilde{\Sigma}\}$$

is called a planar linear system associated to  $\Sigma$ . Let  $\varphi : \mathbb{F}_a \dashrightarrow \mathbb{P}^2$  be the map associated to  $\Lambda_\Sigma$  and  $D \subset \mathbb{F}_a$  an irreducible curve. Then  $(\mathbb{P}^2, \varphi_*(D))$  is called the plane model of  $(\mathbb{F}_a, D)$  associated to  $\Lambda_\Sigma$ .

Every rational map  $\mathbb{F}_a \dashrightarrow \mathbb{P}^2$  sending the general fibre of  $\mathbb{F}_a$  to a line is given by some planar linear system. As a foreword to the next Proposition we want to spend few lines on these linear systems. Let  $\Lambda_\Sigma = \{\Sigma + f_1, \Sigma + f_2, \tilde{\Sigma}\}$  be a planar



linear system on  $\mathbb{F}_a$ , and  $\varphi : \mathbb{F}_a \dashrightarrow \mathbb{P}^2$  the rational map associated. The base locus of  $\Lambda_\Sigma$  is given by

$$\text{Bs } \Lambda_\Sigma = \Sigma \cdot \tilde{\Sigma}$$

and the morphism  $\varphi$  can be factored as follows. Let  $\theta : \mathbb{F}_a \dashrightarrow \mathbb{F}_1$  be the chain of elementary transformation centered  $\text{Bs } \Lambda_\Sigma$ . Then we have  $\theta_*(\Sigma) = C_0 \subset \mathbb{F}_1$ . Let  $\nu : \mathbb{F}_1 \rightarrow \mathbb{P}^2$  be the contraction of the exceptional section  $C_0$ . In this notation we have  $\varphi = \nu \circ \theta$ . In particular given an irreducible curve  $D \subset \mathbb{F}_a$  with  $D \sim \alpha C_0 + \beta f$  then

$$\theta_*(D) \cdot C_0 = D \cdot \Sigma - \sum \mu_i$$

where the  $\mu_i$  are the multiplicities of  $D$  along  $\text{Bs } \Lambda_\Sigma$ . Hence the curve  $\varphi_*(D) \subset \mathbb{P}^2$  has degree

$$(2) \quad \deg \varphi_*(D) = D \cdot \Sigma - \sum \mu_i + \alpha.$$

Our next aim is to single out special planar systems. Consider a model  $(\mathbb{F}_a, \overline{C})$  of Proposition 3.3. Let  $\nu : \mathbb{F}_a \dashrightarrow \mathbb{F}_b$  be a chain of elementary transformations that resolve the singularities of  $\overline{C}$  along  $C_0$ . Let  $\hat{C} := \nu_*(\overline{C}) \sim \alpha C_0 + \beta' f$  be the strict transform. Let  $x_1 \notin C_0$  be a point and  $\text{elm}_{x_1} : \mathbb{F}_b \dashrightarrow \mathbb{F}_{b-1}$  the elementary transformation centered at  $x_1$ . Recursively choose a valuation  $\nu_i$  with centre  $\text{cent}_{\nu_i}(\mathbb{F}_{b-i-1}) = x_i \in \mathbb{F}_{b-i-1} \setminus C_0$  and perform the elementary transformation centered at  $x_i$ . Then after  $b-1$  steps we have

$$\psi_{x_1, \dots, x_{b-1}} := \epsilon \circ \text{elm}_{x_{b-1}} \circ \dots \circ \text{elm}_{x_1} : \mathbb{F}_b \dashrightarrow \mathbb{P}^2$$

where  $\epsilon$  is the blow down of the  $(-1)$ -curve on  $\mathbb{F}_1$ . In particular  $(\psi_{x_1, \dots, x_{b-1}})_*(\hat{C})$  is a curve of degree

$$\hat{C} \cdot (C_0 + (b-1)f) - \sum_1^{b-1} \mu_i + \alpha = \beta' - \sum_1^{b-1} \mu_i$$

where  $\mu_i = \text{mult}_{x_i} \hat{C}$ .

**Definition 3.14.** Let  $(\mathbb{F}_a, \overline{C})$  be a model of Proposition 3.3. Let  $\nu : \mathbb{F}_a \dashrightarrow \mathbb{F}_b$  be a chain of elementary transformations that resolve the singularities of  $\overline{C}$  along  $C_0$ . Let  $\hat{C} := \nu_*(\overline{C})$  be the strict transform. Let  $\{x_1, \dots, x_{b-1}\}$  be a set of centres of valuations as in the above construction. We say that  $\{x_1, \dots, x_{b-1}\}$  is minimal for  $\hat{C}$  if  $\sum \text{mult}_{x_i} \hat{C}$  is maximal. Consider the planar linear system on  $\mathbb{F}_b$

$$\Lambda_{C_0} := \{(C_0 + \sum_1^{|b-1|} f_i) + f_0, (C_0 + \sum_1^{|b-1|} f_i) + f_b, \tilde{\Sigma}\}$$

where  $\tilde{\Sigma} \cdot f_i = x_i$ , for  $i = 1, \dots, b-1$ . Then the plane model associated to  $\Lambda_{C_0}$  will be called a minimal plane model for  $(\mathbb{F}_a, \overline{C})$ .

*Remark 3.15.* Unfortunately the definition of minimal plane models requires a partial resolution of singularities. This cannot be avoided. To convince yourself it is enough to consider a standard model with a unique, nested, singularity along  $C_0$ . The hidden contribution coming from the singularity could drop the degree much more than the smooth points outside.

With this in mind we are ready for the following

**Theorem 3.16.** *Let  $C \subset \mathbb{P}^2$  be a curve, with multiplicity set  $\{m_i\}$ . The curve  $C$  is a minimal degree curve if and only if either  $m_1 < d/3$  or it is a minimal plane model or it is a line.*

*Proof.* Let  $(\mathbb{P}^2, C_{\min})$  be a minimal degree curve. Assume that  $C_{\min}$  is not a line and  $m_1 \geq d/3$ . As noted in remark 3.12 a standard model is a ruled surface. Let  $(\mathbb{F}_a, \overline{C})$  be a standard model for  $(\mathbb{P}^2, C_{\min})$ . Let  $\nu : \mathbb{F}_1 \rightarrow \mathbb{P}^2$  be the blow up of a point  $p \in C_{\min}$  of maximal multiplicity. Let  $\tilde{C} = \nu_*^{-1}(C_{\min})$  be the strict transform, and  $E \subset \mathbb{F}_1$  the exceptional section.

Let  $\overline{C} \sim \alpha C_0 + \beta f$ , and assume  $\kappa(\mathbb{F}_a, 2/\alpha \overline{C}) = 1$ . Then, by Lemma 3.10 (ii), there is a sequence of elementary transformations  $\Phi : \mathbb{F}_1 \dashrightarrow \mathbb{F}_a$ , leading  $(\mathbb{F}_1, \tilde{C})$  to  $(\mathbb{F}_a, \overline{C})$ . In particular  $\Phi_*(E) =: \Sigma$  is a section. Let  $\chi : \mathbb{F}_a \dashrightarrow \mathbb{F}_b$  be a resolution of the singularities of  $\overline{C}$  along  $C_0$ , with  $\hat{C} = \chi_*(\overline{C}) \sim \alpha C_0 + \beta' f$ . Then the linear system taking  $(\mathbb{F}_b, \hat{C})$  to  $(\mathbb{P}^2, C_{\min})$  is a planar linear system associated to  $\Sigma$ , say

$$\Lambda_{\Sigma} = \{\Sigma + f_1, \Sigma + f_2, \tilde{\Sigma}\}.$$

Let  $\Sigma \sim C_0 + \gamma f$  be the section. Then we have

$$\Sigma \cdot \tilde{\Sigma} = 2\gamma + 1 - b,$$

and, by equation (2), the plane model associated to  $\Lambda_{\Sigma}$  has degree

$$(3) \quad \delta_{\Sigma} = \Sigma \cdot \hat{C} - \sum_1^{2\gamma+1-b} \mu_i + \alpha = \alpha(\gamma - b + 1) + \beta' - \sum_1^{2\gamma+1-b} \mu_i$$

where the  $\mu_i$  are multiplicities of  $\hat{C}$  at the centres of elementary transformations. To prove the theorem we have to show that the minimal degree is attained by  $\Sigma_{C_0} = C_0 + (b-1)f$ .

We have to check that

$$-\sum_1^{b-1} \mu_i^0 \leq \alpha(\gamma - b + 1) - \sum_1^{2\gamma+1-b} \mu_i$$

where  $\mu_i^0$  are the multiplicities associated to the planar linear system  $\Lambda_{\Sigma_{C_0}}$ . Note that at least  $b$  points of  $\Sigma \cdot \tilde{\Sigma}$  are outside the exceptional section, there is a unique curve with negative self intersection on  $\mathbb{F}_1$ . Moreover  $(\mathbb{F}_a, 2/\alpha \overline{C})$  is a standard model therefore there are at most  $b - a$  points of multiplicity strictly greater than  $\alpha/2$ , and the corresponding valuations are centered outside  $C_0$ . That is we can assume that

$$\sum_1^{b-1} \mu_i^0 \geq \sum_1^{b-1} \mu_i$$

Hence it is enough to check that

$$\alpha(\gamma - b + 1) - \sum_b^{2\gamma+1-b} \mu_i \geq 0$$

Moreover, we have that

$$\mu_i \leq \frac{\alpha}{2} \quad \text{for } i > b - a$$

then

$$\alpha(\gamma - b + 1) - \sum_b^{2\gamma+1-b} \mu_i \geq \alpha(\gamma - b + 1) - 2(\gamma - b + 1)\alpha/2 = 0.$$

Assume that  $\kappa(\mathbb{F}_a, 2/\alpha\overline{C}) = 0$  then  $a \leq 2$ . Let  $(\mathbb{P}^2, C_{\min})$  be a minimal degree curve birational to  $(\mathbb{F}_a, \overline{C})$ .

*Claim.* A standard model of  $(\mathbb{P}^2, C_{\min})$  is obtained with at most one elementary transformation.

*Proof of the Claim.* Let  $\nu : \mathbb{F}_1 \rightarrow \mathbb{P}^2$  be the blow up of  $p_1 \in C_{\min} \subset \mathbb{P}^2$  and assume that  $(\mathbb{F}_1, \nu_*^{-1}(C_{\min}))$  is not the model. If we have either  $p_2 \notin C_0$  or  $p_3 \notin C_0$ . Then minimal degree forces  $\mathbb{F}_a$  to be either  $\mathbb{F}_0$  or  $\mathbb{F}_2$  with  $m_3 \leq (d - m_1)/2$ , and the claim is clear.

Assume  $p_2, p_3 \in C_0$ , and  $m_3 \geq (d - m_1)/2$ . The requirement on Kodaira dimension forces every model to have  $a \leq 2$ . Therefore, after eventually the first, any elementary transformation centered on the exceptional section has to be balanced by an elementary transformation centered outside the exceptional section. If this chain of elementary transformations take us back onto  $\mathbb{F}_2$  with a curve  $\tilde{C} \subset \mathbb{F}_2$  then

$$\tilde{C} \sim (d - m_1)C_0 + (d + \sum_1^s ((d - m_1) - \mu_j^0) - \sum_1^{s-1} \mu_h)f$$

where  $\mu_j^0$  and  $\mu_h$  are the multiplicity of points on  $C_0$ , respectively, outside  $C_0$ . The standard model construction yields  $\mu_j^0 \geq (d - m_1)/2$  and  $\mu_h > (d - m_1)/2$ . If  $(\mathbb{F}_2, 2/\alpha\tilde{C})$  is not terminal the projection from a non terminal point produces a curve of degree

$$\tilde{d} \leq d + \sum_1^s ((d - m_1) - \mu_j^0) - \sum_1^{s-1} \mu_h - (d - m_1)/2 < d$$

If  $(\mathbb{F}_2, 2/\alpha\tilde{C})$  is terminal then it is the standard model and  $\tilde{C} \cdot C_0 = 0$ . In particular  $\mu_h = d - m_1$ , for any  $h = 1, \dots, s - 1$ , hence a general projection of  $\tilde{C}$  onto  $\mathbb{P}^2$  has degree

$$\tilde{d} = d + \sum_1^s ((d - m_1) - \mu_j^0) - (s - 1)(d - m_1) < d.$$

□

If  $(\mathbb{F}_a, \overline{C})$  is the unique model the claim is enough to conclude.

Assume that  $(\mathbb{F}_a, \overline{C})$  is birational to a model  $(\mathbb{F}_b, \tilde{C})$ . Then we can assume that  $a = 1$  and  $b = 2$ , and one is a standard model of  $(\mathbb{P}^2, C_{\min})$ . The vanishing of Kodaira dimension and  $\alpha$  uniquely determine the linear equivalence class of  $\overline{C}$  and  $\tilde{C}$ , namely

$$\overline{C} \sim \alpha C_0 + \frac{3}{2}\alpha f \text{ and } \tilde{C} \sim \alpha C_0 + 2\alpha f$$

Note further that the existence of two models forces the presence of a canonical singularity. Therefore the minimal plane models obtained by the two have equal degree  $\frac{3}{2}\alpha$ .

In the other direction if  $(\mathbb{P}^2, C)$  has  $m_1 < d/3$  it is a minimal degree curve by Lemma 2.2. Assume that  $(\mathbb{P}^2, C)$  has  $m_1 \geq d/3$  it is not a line and it is a minimal plane model. Let  $(\mathbb{F}_a, \overline{C})$  be a standard model, with  $\overline{C} \sim \alpha C_0 + \beta f$ . If  $\kappa(\mathbb{F}_a, 2/\alpha\overline{C}) = 1$  then Lemma 3.10 and Definition 3.14 allow to conclude. If  $\kappa(\mathbb{F}_a, 2/\alpha\overline{C}) = 0$  then, as we already observed, the linear equivalence class of  $\overline{C}$ ,

and the degree of any minimal plane model is uniquely determined by  $\alpha$ . Proposition 3.6 is therefore enough to conclude.  $\square$

The main question is: can we describe minimal plane model curves (without going through a partial resolution)? We do not expect to have a positive answer in general. In the positive direction there is a nice result of Jung, we are able to recover.

**Corollary 3.17** ([Ju]). *Let  $C \subset \mathbb{P}^2$  be a curve with  $m_1 + m_2 + m_3 \leq d$ . Then  $C$  is a minimal degree curve.*

*Proof.* The assumption forces  $m_i \leq (d - m_1)/2$ . Therefore a minimal plane model is reached by performing the inverse of a resolution of singularities along  $C_0$ .  $\square$

Unfortunately the opposite direction is not true, even discarding the trivial example of lines.

**Example 3.18.** *Let  $C \subset \mathbb{P}^2$  be a curve of degree 7 with a point of multiplicity 4 and two infinitely near double points. Then it is easily seen that  $C$  is a minimal curve.*

The difficulty in predicting minimality for plane curves can be seen in the following example.

**Example 3.19.** *Let  $D_i \subset \mathbb{F}_3$  be irreducible and reduced curves with  $D_i \sim 3C_0 + 11f$ . Assume that  $D_1$  has a unique ordinary double point, say  $p_1 \in C_0$  and  $D_2$  has a unique ordinary double point, say  $p_2 \in \mathbb{F}_3 \setminus C_0$ . The pair  $(\mathbb{F}_3, D_i)$  is birational to a pair  $(\mathbb{P}^2, C_i)$ . Where  $C_i$  is a degree 9 curve with a six-tuple point and 3 infinitely near double points. The main difference, that we can only divine on the resolution, is that  $C_1$  is a minimal degree curve, while  $C_2$  is birational to a curve of degree 8 with a quintuple point and an infinitely near double point.*

It is easy to produce similar examples of arbitrarily nested singularities.

#### 4. RATIONAL DIVISORS

It is natural to ask when a rational hypersurface is Cremona equivalent to a hyperplane. The unique case in which an answer is known is that of rational curves. Coolidge, [Co], first suggested that this should be the case if and only if  $\bar{\kappa}(\mathbb{P}^2, C) < 0$ . For any pair  $(X, D)$  we will indicate with  $\bar{\kappa}(X, D)$  the log-Kodaira dimension of a log resolution. Kumar–Murthy were able to prove the following, see also [Li, Proposition 12].

**Theorem 4.1.** [KM] *A rational plane curve is Cremona equivalent to a line if and only if  $|2K_{\mathbb{P}^2} + C| = \emptyset$ .*

It is interesting to give a LMMP approach to this statement.

**Proposition 4.2.** *A rational plane curve  $C$  is Cremona equivalent to a line if and only if  $\bar{\kappa}(\mathbb{P}^2, 1/2C) < 0$ .*

*Proof.* We have only to prove that if  $\bar{\kappa}(\mathbb{P}^2, 1/2C) < 0$  then  $C$  is Cremona equivalent to a line. Let  $g : S \rightarrow \mathbb{P}^2$  be a minimal resolution of singularities, with  $C_S = g_*^{-1}(C)$ . Then by hypothesis we have  $\kappa(S, 1/2C) < 0$ . Let us start a LMMP

program for the pair  $(S, 1/2C_S)$ . The hypothesis on Kodaira dimension shows that every  $(-1)$ -curve contracted during the LMMP satisfies

$$(K + 1/2\overline{C}) \cdot E < 0$$

where  $K$  is the canonical class and  $\overline{C}$  the strict transform of the curve  $C$ . This means  $C \cdot E < 2$ . In particular the output of the LMMP is a pair  $(\overline{S}, 1/2\overline{C})$  with  $\overline{C}$  smooth and  $K_{\overline{S}} + 1/2\overline{C}$  negative on infinitely many curves. If  $\overline{S}$  is  $\mathbb{P}^2$  the claim is clear. Otherwise  $\overline{S}$  is a rational ruled surface and  $\overline{C}$  is a smooth rational curve. A simple computation allows to conclude that  $\overline{C}$  is one of the following:

- a fibre of one ruling in  $\overline{S}$ ,
- a section of one of the ruling in  $\overline{S}$ ,
- $\overline{C} \sim 2(C_0 + f)$  and  $\overline{S} \cong \mathbb{F}_1$ .

In all these cases  $(S, \overline{C})$  is easily seen to be birational to a line in  $\mathbb{P}^2$ .  $\square$

The case of plane curves has been studied from many different points of view. Various other necessary and sufficient conditions are known, see [FLMN] for a nice survey. Probably the most tempting conjecture is Nagata–Coolidge’s prediction that every cuspidal rational curve is Cremona equivalent to a line, [Co] [Na]. For this we do not see any translation into LMMP dictionary.

It is quite natural to ask for generalisations in higher dimensions. The main difficulty is the poor knowledge of Cremona group starting from  $\mathbb{P}^3$ .

The case of surfaces is already quite mysterious. It is easy to show that quadrics and cubics are Cremona equivalent to a plane, see Case 4.16 below. Rational quartics with either 3-ple or 4-uple points are again easily seen to be Cremona equivalent to planes, the latter are cones over rational curves Cremona equivalent to lines.

It has been expected that Noether quartic should be the first example of rational surface not Cremona equivalent to a plane, but this is not the case.

**Example 4.3.** *Let  $S \subset \mathbb{P}^3$  be the Noether quartic. That is a quartic with a unique double point of local analytic type  $O \in (x^2 + f_3 + g_4 = 0) \subset \mathbb{C}^3(x, y, z)$ . We can assume that the equation of  $S$  is*

$$(x_0^2 x_3^2 + f_3 x_3 + g_4 = 0) \subset \mathbb{P}^3$$

*with  $p \equiv [0, 0, 0, 1] \in S$  the unique singular point. Let  $\epsilon : Y \rightarrow \mathbb{P}^3$  be a weighted blow up of  $p$ , with weights  $(2, 1, 1)$ , and exceptional divisor  $E$ . Then  $\epsilon^*(x_0 = 0) = H + 2E$  and we can contract  $H$  to a curve. Let  $\mu : Y \rightarrow V$  be the contraction of  $H$ . Then  $V$  is the cone over the Veronese surface in  $\mathbb{P}^6$ . The linear system providing this birational map is that of quadric with multiplicity 2 on the valuation  $E$ . Therefore we have  $S_V := (\mu \circ \epsilon)_*(S) \in |\mathcal{O}_V(2)|$ , and  $S_V$  is a smooth surface in  $V$ .*

*Let  $A_1$  be a general hyperplane section of  $V$  and consider  $\Gamma := A_1 \cap S_V$ , a smooth irreducible curve. Let  $C \subset S_V$  be a general smooth conic, and*

$$\mathcal{B} = \{B_0, B_1, B_2, B_3\} = |\mathcal{O}_V(1) \otimes \mathcal{I}_C|.$$

*Let us consider the linear system*

$$\mathcal{C} = \{S_V, A_1 + B_0, A_1 + B_1, A_1 + B_2, A_1 + B_3\} \subset |\mathcal{O}_V(2)|$$

*and  $\varphi : V \dashrightarrow \mathbb{P}^4$  the rational map associated. Then*

$$S_V \cdot (A_1 + B_i) = \Gamma + C + R_i$$

for some residual curve  $R_i \subset B_i$ . Let  $v : \mathbb{P}^2 \rightarrow B_i$  be the Veronese embedding. In this notations  $v^*(C)$  is a line,  $v^*(S_{V|B_i}) \sim \mathcal{O}(4)$ . Therefore the residual curve  $R_i$  is the image of a cubic curve via the Veronese map  $v$ . Moreover we have  $B_j|_{B_i} = C + D_{ji}$  and  $v^*(D_{ji}) \sim \mathcal{O}(1)$ . Hence  $Z_{ij} := R_i \cdot B_j$  is a zero dimensional scheme of length 6 with only 3 points outside of  $C$ . This means that  $\varphi(V)$  is a, singular, cubic hypersurface and therefore a projection from a singular point gives the required Cremona.

Note that for the Noether quartic we have  $\kappa(\mathbb{P}^3, S) < 0$ . This example and the proof of Proposition 4.2 suggest that a result similar to Proposition 4.2 is at hand also for  $\mathbb{P}^3$ . To get it we have to slightly modify the  $\sharp$ -MMP developed in [Me] for linear systems on uniruled 3-folds.

**Definition 4.4** ([Me]). Let  $T$  be a terminal  $\mathbb{Q}$ -factorial uniruled 3-fold and  $H$  an irreducible and reduced nef Weil divisor on  $T$ . Let

$$\rho = \rho_H = \rho(T, H) =: \sup \{m \in \mathbb{Q} | H + mK_T \text{ is an effective } \mathbb{Q}\text{-divisor} \} \geq 0,$$

be the threshold of the pair  $(T, H)$ .

*Remark 4.5.* Note that the threshold is not a birational invariant of the pair. Think for instance to  $(\mathbb{P}^3, H)$ , with  $H$  a plane, and  $(\mathbb{P}^3, \mathbb{Q})$ . On the other hand  $\rho < 1$  is clearly equivalent to  $\bar{\kappa}(T, H) < 0$ . Therefore the bound  $\rho < 1$  is a birational invariant.

The main result we need in this contest is the following modification of [Me, Theorem 5.3].

**Theorem 4.6.** *Let  $(\mathbb{P}^3, S)$  be a pair with  $S$  a rational surface and  $\bar{\kappa}(\mathbb{P}^3, S) < 0$ . Then  $(\mathbb{P}^3, S)$  is birational to  $(T, S_T)$  where the pair  $(T, S_T)$  is one of the following:*

- i) *a rational  $\mathbb{Q}$ -Fano 3-fold  $T$  of index  $1/\rho > 1$ , with  $K_T \sim -1/\rho S_T$ , of the following type:*
  - a)  $(\mathbb{P}(1, 1, 2, 3), \mathcal{O}(6))$
  - b)  $(X_6 \subset \mathbb{P}(1, 1, 2, 3, a), X_6 \cap \{x_4 = 0\})$ , with  $3 \leq a \leq 5$
  - c)  $(\mathbb{P}(1, 1, 1, 2), \mathcal{O}(2k))$ , with  $k = 1, 2$ ,
  - d)  $(X_4 \subset \mathbb{P}(1, 1, 1, 2, a), X_4 \cap \{x_4 = 0\})$ , with  $2 \leq a \leq 3$
  - e)  $(\mathbb{P}^3, \mathcal{O}(a))$ , with  $a \leq 3$ ,
  - f)  $(X_3 \subset \mathbb{P}(1, 1, 1, 1, 2), X_3 \cap \{x_4 = 0\})$ ,
  - g)  $(\mathbb{Q}^3, \mathcal{O}(b))$ , with  $b \leq 2$ ,  $X_{2,2} \subset \mathbb{P}^5, \mathcal{O}(1)$ , a linear section of the Grassmann variety parametrising lines in  $\mathbb{P}^4$ , embedded in  $\mathbb{P}^9$  by Plucker coordinates;
- ii) *a bundle over a  $\mathbb{P}^1$  with generic fibre  $(F, S_{T|F}) \cong (\mathbb{P}^2, \mathcal{O}(2))$  and with at most finitely many fibres  $(G, S_{T|G}) \cong (\mathbb{S}_4, \mathcal{O}(1))$ , where  $\mathbb{S}_4$  is the cone over the normal quartic curve and the vertex sits over an hyper-quotient singularity of type  $1/2(1, -1, 1)$  with  $f = xy - z^2 + t^k$ , for  $k \geq 1$ , [YPG];*
- iii) *a quadric bundle over  $\mathbb{P}^1$  with at most  $cA_1$  singularities of type  $f = x^2 + y^2 + z^2 + t^k$ , for  $k \geq 2$ , and  $S_{T|F} \sim \mathcal{O}(1)$ ;*
- iv)  $(\mathbb{P}(E), \mathcal{O}(1))$  *where  $E$  is a rk3 vector bundle over  $\mathbb{P}^1$ ;*
- v)  $(\mathbb{P}(E), \mathcal{O}(1))$  *where  $E$  is a rk2 vector bundle over a rational surface  $W$ ;*
- vi)  $\pi : T \rightarrow W$  *is a Mori fibre space and  $S_T = \pi^*A$  for some effective divisor  $A$  on  $W$ .*

*Proof.* We already observed that  $\bar{\kappa}(\mathbb{P}^3, S) < 0$  is equivalent to  $\rho(\mathbb{P}^3, S) < 1$ . We mimic the proof of [Me, Theorem 5.3]. The main difference is that we are assuming that the 3-fold is rational and  $S$  is not a big linear system but a fixed divisor. Let  $\nu : Y \rightarrow \mathbb{P}^3$  be a log resolution of the pair  $(\mathbb{P}^3, S)$ . Let  $S_Y = \nu_*^{-1}S$  be the strict transform. First we have to check that  $S_Y$ , and its direct images, are not contracted along the  $\#$ -MMP. To do this observe that the  $\#$ -MMP is directed by the nef value

$$\sup\{m \in \mathbb{Q} \mid mK_Y + S_Y \text{ is nef}\},$$

and every birational modification is centered on an extremal ray, i.e. the canonical class is relatively antiample. Let  $C$  be a curve contracted along the  $\#$ -MMP. Then we have  $(mK_Y + S_Y) \cdot C = 0$ , for some  $m \geq 0$  and  $K_Y \cdot C < 0$ . Therefore  $S_Y \cdot C \geq 0$  and the surface  $S_Y$  cannot be contracted by the steps of the program. If  $\rho(\mathbb{P}^3, S) > 0$  the list is just the one of [Me, Theorem 5.3] with  $T$  rational. In particular for Fanos it is enough to amend the list in [Me] using the list of non rational smooth Fanos in [Is, Theorem 5]. This gives all cases from i) through v). If  $\rho(\mathbb{P}^3, S) = 0$  then  $S_T$  is relatively trivial and we end up in case vi).  $\square$

Let us start to prove some Cremona equivalences.

**Lemma 4.7.** *Assume that  $(T, S_T)$  is in cases ii)–v) of the above list. Then it is Cremona equivalent to a plane in  $\mathbb{P}^3$ .*

*Proof.* We treat the different possibilities separately.

*Case 4.8 (ii).* Let  $\eta$  be the general point of the base  $\mathbb{P}^1$ . Consider  $F_\eta \cong \mathbb{P}_\eta^2$ , the generic fibre of  $T$ , and  $C_\eta$  the generic fibre of  $S_T$ , a conic. Then by Tsen's theorem  $C_\eta$  has many rational points. Choose three general points on  $C_\eta$  and do a standard Cremona transformation on  $F_\eta$  centered at these points. This sends  $F_\eta$  to  $\mathbb{P}_\eta^2$  and  $C_\eta$  to a line  $l_\eta$ . This induces a birational modification on  $T$  and shows that  $(T, S_T)$  is birational to a pair in case iv).

*Case 4.9 (iii).* As in the previous case let  $\eta$  be the generic point of the base  $\mathbb{P}^1$ . Then  $F_\eta \cong \mathbb{Q}_\eta \subset \mathbb{P}_\eta^3$  is the generic fibre of  $T$  and  $C_\eta$  is again a conic with many points. The projection from a point sends  $F_\eta$  to  $\mathbb{P}_\eta^2$  and  $C_\eta$  to a line  $l_\eta$ . Therefore, also in this case the pair  $(T, S_T)$  is birational to a pair in case iv).

*Case 4.10 (iv).* Via elementary transformations centered in a bunch of points we can modify  $(T, S_T)$  into the pair  $(\mathbb{P}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1)), \mathcal{O}(1))$ . This shows that  $(T, S_T)$  is birational to a surface  $S \subset \mathbb{P}^3$  of degree  $d$  with a line of multiplicity  $d - 1$ . Let  $l \subset S$  be this line. To conclude these cases we prove by induction on the degree  $d$  that  $S$  is Cremona equivalent to a plane. The case  $d = 2$  is immediate. Let  $\omega : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$  be the  $T_{2,3}$  map, given by the linear system

$$|\mathcal{O}(2) \otimes \mathcal{I}_l \otimes_1^3 \mathcal{I}_{p_i}|$$

where  $p_i$  are general points of  $S$ . Let  $l'$  be the image of the plane  $\langle p_1, p_2, p_3 \rangle$  via  $\omega$ , keep in mind Construction 1.4. Then  $\omega(S) \subset \mathbb{P}^3$  is a surface of degree

$$3d - 2(d - 1) - 3 = d - 1$$

with the line  $l'$  of multiplicity  $2d - (d - 1) - 3 = d - 2$ . We conclude by induction hypothesis.

*Case 4.11 (v).* Let  $D \subset T$  be a general very ample divisor, and

$$\Gamma := S_T \cap D$$

the smooth intersection, keep in mind that  $S_T$  is smooth. Let  $\text{elm}_\Gamma : T \dashrightarrow T_1$  be the elementary transformation centered on  $\Gamma$ . Let  $S_1 := \text{elm}_\Gamma(S)$ , then as we observed in Construction 1.7  $T_1$  is still a scroll over the rational surface  $W$ , and  $S_1 \cong W$ . Let  $\nu : W \dashrightarrow \mathbb{P}^2$  be a birational map. Then, as explained in [Me, 5.7.4], we can follow this birational map on the 3-fold. This produces a birational map  $\phi : T_1 \dashrightarrow T'$  and diagram

$$\begin{array}{ccc} T_1 & \xrightarrow{\phi} & T' \\ \pi \downarrow & & \downarrow \pi' \\ W & \xrightarrow{\nu} & \mathbb{P}^2 \end{array}$$

In this way we end up with a pair  $(T', S')$ , birational to  $(T, S_T)$ , with  $T'$  a scroll over  $\mathbb{P}^2$  and  $S' \cong \mathbb{P}^2$  a section. Then  $S'_{|S'} \sim \mathcal{O}(s)$  for some  $s \leq 0$ . If  $s < 0$  consider a curve  $C \subset T'$  with  $\pi'_*(C) \sim \mathcal{O}(s)$  and  $C \cap S' = \emptyset$ . Then  $\text{elm}_C$  modify  $(T', S')$  into  $(\mathbb{P}^2 \times \mathbb{P}^1, F)$ , with  $F = p^*(\mathcal{O}_{\mathbb{P}^1}(1))$ . To conclude it is then enough to project from two general points of this 3-folds in its Segre embedding.

□

*Remark 4.12.* A similar treatment for case vi) seems out of reach. The main difficulty comes from rational conic bundles, that are not scrolls. The knowledge of these 3-fold is very poor. Even assuming the standard conjectures, like Cantor or Iskovskikh rationality criteria, it is very difficult to guess any kind of birational embedding of  $S_T$  in  $\mathbb{P}^3$ . This difficulty forces us to ask for a positive threshold in the main theorem below.

It is clear that if  $S$  is Cremona equivalent to a plane then  $\bar{\kappa}(\mathbb{P}^3, S) < 0$ . We want to prove that the converse is also true, under a mild assumption.

**Theorem 4.13.** *Let  $S \subset \mathbb{P}^3$  be an irreducible and reduced surface, assume that  $\rho(\mathbb{P}^3, S) > 0$ . Then  $S$  is Cremona equivalent to a plane if and only if  $\bar{\kappa}(\mathbb{P}^3, S) < 0$ .*

*Proof.* Thanks to Theorem 4.6, Lemma 4.7 and the assumption on the threshold we have only to worry about Fanos. Unfortunately we do not have a general argument. To conclude we have to produce for any variety in the list a birational map to  $\mathbb{P}^3$  that sends  $S_T$  to a surface we are able to treat. Let  $(T, S_T)$  be the pair birational to  $(\mathbb{P}^3, S)$ .

*Case 4.14 (a, b).* Assume first  $(T, S_T) = (\mathbb{P}(1, 1, 2, 3), \mathcal{O}(6))$ . Let  $\epsilon : Y \rightarrow \mathbb{P}^3$  be the weighted blow up of the point  $[0, 0, 0, 1]$  with weights  $(1, 2, 3)$ , on the coordinates  $(x_0, x_1, x_2)$ , and exceptional divisor  $E$ . We choose the coordinate in such a way that the plane  $H = (x_2 = 0)$  satisfies

$$\epsilon^*H = H_Y + 3E,$$

that is  $\epsilon|_{H_Y} : H_Y \rightarrow H$  is the weighted blow up  $(1, 2)$ . In particular on  $Y$  there is a line  $l$ , the strict transform of  $(x_1 = x_2 = 0)$ , with normal bundle

$$N_{l/Y} = \mathcal{O}(-1) \oplus \mathcal{O}(-2)$$



Let  $\mu : Y \dashrightarrow Y'$  be the antiflip of  $l$ , then  $H' = \mu_* H_Y = \mathbb{P}(1, 1, 2)$  is a cone with the vertex over a terminal point of type  $1/2(1, 1, -1)$  and we can blow it down to a smooth point, with a morphism  $\nu : Y' \rightarrow Z$ . Let us understand what is  $Z$ . We have  $\text{rk Pic}(Z) = 1$ . Let  $\Lambda \subset |\mathcal{O}_{\mathbb{P}^3}(1)|$  be the pencil of hyperplanes through  $l$ ,  $\varphi := \nu \circ \mu \circ \epsilon$ , and  $\varphi_* \Lambda = \mathcal{A}$ . Then we have

$$-K_Y = \epsilon^*(-K_X) - 5E = 4H + (12 - 5)E$$

moreover  $A := (\nu \circ \mu)_* E \in \mathcal{A}$  and this yields

$$-K_Z \sim 7A$$

By construction  $6E$  is a Cartier divisor. Therefore we find a Fano 3-fold of index greater than 1 and it is easy to realize that  $Z \cong \mathbb{P}(1, 1, 2, 3)$ , and  $A \sim \mathcal{O}(1)$ . Via the map  $\varphi$  we can also easily understand the elements of  $|\mathcal{O}_Z(6)|$ . Let  $F \subset |\mathcal{O}_{\mathbb{P}^3}(3)|$  be a cubic surface with  $\text{mult}_E(F) = 3$ . Then

$$-K_Y = \epsilon^*(F + H) - 5E = F_Y + H_Y + E$$

hence we get

$$7A = -K_Z = (\nu \circ \mu)_*(F_Y + H_Y + E) = F_V + A,$$

where  $F_V = \varphi_*(F) \in |\mathcal{O}_Z(6)|$ . Equivalently  $S_T \in |\mathcal{O}_Z(6)|$  is birational to a cubic and hence to a plane. The pair in b) projects birationally, from the singular point, to the pair in a).

*Case 4.15 (c, d).* The pair  $(\mathbb{P}(1, 1, 1, 2), \mathcal{O}(2))$  is the cone over the Veronese surface, a minimal degree 3-fold, with an hyperplane section. It projects to  $(\mathbb{P}^3, \mathcal{O}(2))$  from two general points in  $S_T$ . The pair  $(\mathbb{P}(1, 1, 1, 2), \mathcal{O}(4))$  has been discussed in Example 4.3. The pairs in d) projects birationally, from the singular point, to  $(\mathbb{P}(1, 1, 1, 2), \mathcal{O}(4))$ .

*Case 4.16 (e, f, g).* The quadrics in  $\mathbb{P}^3$  are easy, think for instance to  $T_{2,3}$ . Cubics with isolated singularities have been treated in Construction 1.5. Cubics with non isolated singularities are singular along a line and we can apply the proof of case 4.10 in Lemma 4.7. The pair in f) projects, birationally, from the singular point to  $(\mathbb{P}^3, \mathcal{O}(3))$ . Similarly all the pairs in g) project, from a suitable number of points in  $S_T$ , to a pair in e).

□

*Remark 4.17.* In arbitrary dimensions a weaker statement can be obtained without much effort. Let  $(\mathbb{P}^k, D)$  be a pair and assume that  $0 < \rho(\mathbb{P}^k, D) < \frac{1}{k-2}$ . Then following [Me, Corollary 5.10], thanks to the existence of MMP in arbitrary dimension, [BCHM], we end up in the usual list and it is easy to conclude as before.

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